Grading guide, Pricing Financial Assets, June 2015

1. Let the price of a traded financial instrument, S, be modelled (under the probability measure $\mathbb{P}$) by the geometric Brownian motion

$$dS = \mu S dt + \sigma S dz$$

where μ and $\sigma > 0$ are constants, and where dt and dz are the standard short hand notations for a small time-step and a Brownian increment.

   (a) Describe the qualitative characteristics of this model, and discuss it's possible shortcomings as a model of a stock price.

   (b) Assume that the instrument pays a continuous dividend stream of q proportional to the price S. What will the drift rate of the price be under the standard risk neutral probability measure ($\mathbb{Q}$) and a no-arbitrage assumption?

   (c) Consider the transformation G of S given by the natural logarithm (ln), i.e. $G(x) = \ln(x)$. Use Ito's lemma to find the process followed by $G(S)$.

   (d) Suppose that for $t = 0$ the price of the instrument is S_0. What is the expectation (under $\mathbb{P}$) of the natural log of the price at $t = T \geq 0$?

Solution:

(a) The answer should discuss

- the drift rate and the volatility
- the continuous sample paths taken by prices
- the distribution of prices and return
- and that these characteristics often are at odds with empirical findings for stock prices, where stock price volatility is not constant, stock prices may jump, and return distributions thus will exhibit fatter tails

(b) Under the risk neutral probability measure the drift should be $r - q$, where r is the short term risk free interest rate (here assumed constant).

(c) The derivation for the $\ln$-transformation is given in Hull, section 13.7.

(d) The expectation

$$E[\ln S_T] = \ln S_0 + \left(\mu - 0.5\sigma^2\right) T$$

follows from the result above.

2. (a) Consider a derivative with price $V(S,t)$ as some function of the current stock price S and time t (and further implicit parameters). Define and interpret the Delta, Gamma and Theta of the derivative.

   (b) Assume that the stock pays no dividends before time T, and that there is a constant risk free interest rate of r. Let $c(S, K, T, r)$ and $p(S, K, T, r)$ be the price at time $t = 0$ of a European call and a European put, respectively, on the stock with the same strike K and expiry T. Derive the call-put-parity.

   (c) Use the call-put-parity to find a relationship between the Deltas of the call and put. Repeat this for Gamma and Theta, respectively.

(d) Suppose a portfolio of the stock and/or derivatives of that stock is Delta-neutral, and that there are no arbitrage possibilities. Let the value of the portfolio be $\Pi(S,t)$. What can we say about the relation between the Theta and Gamma of the portfolio?

Solution:

(a) Cf. Hull p.380ff

(b) By considering the payoff at maturity you see that a portfolio with long call and a short put has the same payoff as a forward on the stock with forward price K, thus also (under a no-arbitrage assumption)

$$c(S,K,T,r) + K\mathsf{e}^{-rT} - p(S,K,T,r) - S = 0$$

(c) By considering the call-put-parity you immediately get for the Deltas

$$\Delta_c = 1 + \Delta_p$$

and using this for the gammas

$$\Gamma_c = \Gamma_p$$

For Theta we get

$$\Theta_c + rK\mathsf{e}^{-rT} - \Theta_p = 0$$

(d) For a portfolio with value $\Pi(S,t)$ dependent on a non-dividend-paying stock we have by no arbitrage the Black-Scholes-Merton PDE:

$$\frac{\partial \Pi}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 \Pi}{\partial S^2} + rS\frac{\partial \Pi}{\partial S} - r\Pi = 0$$

or

$$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta = r\Pi$$

When delta is zero you have

$$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

One interpretation from this is that a delta-neutral, positive-gamma portfolio barring arbitrage will have (for small r) a negative theta (time-decay), cf. Hull, section 18.7.

3. Consider an interest rate floor with a life of T, a principal of N, and a floor rate of R_K. Consider reset dates $0 = t_0 < t_1 < t_2, \ldots < t_n$, and let R_k be the Libor rate for the period from t_k to t_{k+1} known at t_k.

(a) Describe the payments of the floor, and define a floorlet.

(b) Show that the floor can be considered as a portfolio of European call options on zero-coupon bonds (for notational ease you may assume $N = 1$) *(Hint: Start by analyzing a single floorlet)*.

(c) A standard market practice is to price a floorlet (at $t = 0$) with the Black formula:

$$Floorlet^{Black}(0,k,N,R_K,\sigma_k) = NP(0,t_{k+1})\tau_k(R_K\Phi(-d_2) - F_k\Phi(-d_1))$$

with

$$d_1 = \frac{ln(F_k/R_K) + 0.5\sigma_k^2 t_k}{\sigma_k\sqrt{t_k}}$$
$$d_2 = d_1 - \sigma_k\sqrt{t_k}$$

where $P(t,T)$ is the price at t of a zero coupon bond maturing at T, F_k is the forward rate at 0 for the time interval (t_k, t_{k+1}) with length τ_k, and σ_k the volatility of this rate. The function Φ is the standard Normal distribution function. What assumptions can justify this formula?

Solution:

(a)

Definition 0.1 (Floors). *Let the Libor-rate set at t_k for the period (t_k, t_{k+1}) be R_k. Consider a maturity T, a principal of N and a cap rate of R_K. Let reset dates $t_1, t_2, \ldots, t_n$ be given and $t_{n+1} = T$. A Floor is a contract that gives its holder the payments*

$$N\tau_k max(R_k - R_K; 0)$$

at time t_{k+1} (i.e. in arrears), where τ_k is the time length between t_k and t_{k+1}

Hull defines a floor with payments in arrears p.654. The potential payment at each payoff date defines the payment of a single floorlet starting on the previous reset date.

(b) Cf. Hull p. 654 for the case of a cap.

- A floor can equivalently be seen as a portfolio of calls on properly defined Zero Coupon Bonds (ZCBs)

$$\frac{N\tau_k}{1 + R_k\tau_k} max(R_K - R_k; 0) =$$
$$\frac{N\tau_k}{1 + R_k\tau_k} max((1/\tau_k + R_K) - (1/\tau_k + R_k); 0) =$$
$$max(N\frac{1 + R_K\tau_k}{1 + R_k\tau_k} - N; 0)$$

- The first part in the first argument under the max-operator is the value at t_k of a ZCB with payoff $N(1 + R_K\tau_k)$ at t_k; thus the payoff is a call on this particular ZCB with strike N

(c) Cf. Hull p. 657. The application of the Black model is consistent with a world that is forward risk neutral wrt a ZCB maturing at t_{k+1}. Under this probability measure assume R_k has a lognormal distribution with a standard deviation of $\ln(R_k)$ of $\sigma_k\sqrt{t_k}$. We also have under this measure that the expectation of R_k is the forward rate F_k.